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Mathieu's equation, classical mechanical applications, and reduction of complex differential equation systems to simpler forms

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Abstract

The goal of this study is to reduce the rank of differential equations and describe the most efficient way to do this. We will consider the boundary value problem of a nonlinear system of differential equations and provide a systematic overview of the methods for determining the corresponding stability diagram, its structure and properties. Some problems may be difficult to solve directly, especially the Mathieu differential equation problem, which has many useful applications in theoretical and experimental physics. Therefore, a useful way to solve them is to transform nonlinear differential equation systems into a system of linear differential equations. In this study, we will use the solutions of the Mathieu differential equation, the pendulum equation and the parachute equation as examples, and develop a method for solving them. The method includes the analysis of the motion process, the landing speed and the friction coefficient of the average speed. We have improved other factors compared with traditional methods to create a complete method for determining the starting point and the landing point. The study recommends it as a reference for other analysis methods, and we provide clear examples.

Keywords: *Mathematical Modeling, Higher-order Equations, Mathieu Differential Equation, Parachuting, Displacement function*

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1. Introduction

To find approximate solutions to nonlinear differential equations when an exact solution cannot be obtained by conventional methods. In general, for vibration and nonlinear problems, the Matthew equation in its classical form is necessary to transform it into a linear differential equation to facilitate its solution. How is it solved after it has been reduced? It involves the use of numerical methods. These methods create successive approximations that converge to the exact solution of the equation or system of nonlinear differential equations, ultimately reducing its degree. However, while our approach focused on solving higher-order nonlinear differential equations in several variables involving only one variable, it only addressed solving nonlinear differential equations in a single variable, rather than those involving multiple variables (*Ascher, U.M. and L.R. Petzold, 1998: SIAM.*). In many applications, it is necessary to have a comprehensive understanding of the dynamics of complex structures that use differential equations such as helicopters, airplanes, buildings, bridges, and vehicles. While modern design tools such as finite element analysis have greatly expanded the mathematical modeling available for such models, they are often limited in their dynamic capabilities, especially when the dynamics of the structure introduces a nonlinear system of differential equations (*Ascher, U.M. and L.R. Petzold, 1998: SIAM*.). Previous studies have addressed this topic such as Reduction of Nonlinear Equations in Mathematical Mechanics and Physics, New Integral Equations and Exact Solutions (*Antontsev, S.N., 2002*), and The Mathieu differential equation problem and Its Generalizations: An Overview of Stability Diagrams and Their Properties (*Kovacic, I.,R.Rand, and S. Mohamed Sah,2018.*). The main objective of this study is to address and develop a model for reducing the order of differential equations by simplifying the differential equations to ordinary linear equations. The study model developed by (*Semler, C., W. Gentleman, and M. Paıdoussis, 1996., Shakeri, F. and M. Dehghan, 2008., Stamenković, M., 2012.)* extends to obtaining exact models of higher-order equations and then reducing them to systems of linear differential equation models. Mathieu's differential equation problem finds wide applications in many areas of classical mechanics, including describing the vibrations of mechanical systems with time-varying parameters, such as the vibrations of bridges and viaducts under the influence

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of wind or earthquakes. Studying the stability of dynamic systems and determining the conditions that lead to the occurrence of linear and nonlinear resonance phenomena *(Stamenković, M.,2012)*.Describing the motion of charged particles in a time-varying electromagnetic field. Studying the motion of celestial bodies under the influence of changing gravitational forces.

2. **Study Problem**

The fundamental problem in studying the nonlinear differential equation of Mathieu's differential equation problem and its applications to classical mechanics is how can we reduce complex systems of nonlinear differential equations to linear differential equations. The importance of the study in solving the fundamental problem lies in the fact that despite the simplicity of the form of the equation, the complexity can be reduced by converting systems of nonlinear differential equations to linear systems to make the system more amenable to analysis and understanding. *(Bruno, A.D., 2000).* How can the Mathieu's differential equation problem and its very complex solutions be made simpler, more amenable and flexible? Reducing systems of differential equations to their simplest forms is a fundamental goal in many scientific and engineering fields. This simplification allows for a better understanding and analysis of the behavior of the system. (*Semler, C., W. Gentleman, and M. Paıdoussis, 1996*.). The main problem in studying the Mathieu's differential equation problem is that there are no general analytical solutions to this equation, except in special and simple cases (*Shakeri, F. and M. Dehghan, 2008.*). This means that the solutions of a nonlinear differential equation cannot be expressed in terms of known elementary functions (such as exponential and trigonometric functions), making it difficult to analyze and predict the behavior of the solutions. Reducing complex systems of nonlinear differential equations to simpler forms The Mathieu's differential equation problem is an important starting point in reducing complex systems of differential equations to simpler forms. In many cases, the behavior of complex systems can be approximated using the Matthew equation, making them easier to analyze and study (*Chaturantabut, S. and D.C. Sorensen. 2009.*). Natural pattern method the system is analyzed into simple natural patterns, and each pattern is described by a separate Matthew equation.

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Averaging method the time averages of the periodic coefficients in the equation are calculated, resulting in an equation with constant coefficients that is easy to solve. Numerical approximation methods: Using computers to process the equation and calculate the solutions approximately. Importance of studying the Mathieu's differential equation (*Ibragimov, N.K., 1992*.).

3. **Study Questions**

The main question that the researcher asks is:

How can we reduce complex systems of nonlinear differential equations to linear differential equations.

There are some sub-questions:

1. How can we speed up the simulations and calculations needed to analyze the system?

2. How can we solve the basic problem that despite the complex form of the equation, the complexity can be reduced by converting systems of nonlinear differential equations to linear systems to make the system more analyzable and understandable.

3. How can we find general analytical solutions to Mathieu's differential equation problem that can be generalized even though there are no general analytical solutions to this equation, except in special and simple cases

4. Study Significance

The Mathieu's differential equation problem is a linear differential equation with periodic coefficients and describes a wide range of physical and engineering phenomena. Despite its apparent simplicity, it carries great mathematical complexities and is considered one of the most important differential equations in the field of analytical mechanics and vibration theory. The Mathieu's differential equation problem finds wide applications in many areas of classical mechanics, including It is an important starting point in reducing complex systems from differential equations to simpler forms. In many cases, the behavior of complex systems can be

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approximated using the Matthew equation, which facilitates their analysis and study. Simplifying nonlinear differential equation systems into linear differential equations helps in predicting the behavior of complex systems, which contributes to improving their design and maintenance. The differential equation is a powerful tool for understanding and analyzing many physical and engineering phenomena. Despite the challenges facing its study, ongoing research contributes to the development of new methods for solving and analyzing it, which opens new horizons for applications in various fields.

5. Methodology

The study relied on the analytical research approach using MATLAB program based on analytical methods and used numerical analysis and algorithms to reach the results and followed the exploratory analytical research approach through the results of the algorithms (*Ibragimov, N.K., 1992*.). Based on previous studies in calculations and applications of Mathieu's functions from a historical perspective, (*Hide, R., 1997*.) Mathieu's differential random equation, nonlinear differential equations, and dynamic systems, (*Stamenković, M.,2012*., *Teschl, G.,.2012*). Exact solutions to Matthew's equation, (*Daniel, D.J., 2020.* ***2020***). The main objectives of this study are as follows:

• Develop more accurate and efficient mathematical models to describe the behavior of complex and nonlinear systems.

• Use the results obtained in various fields such as mechanical engineering and others

• Reduce complexity by converting nonlinear differential equation systems into linear systems to make the system more analyzable and understandable. And speed up the simulation and calculation processes required to analyze the system.

6. Converting a general higher-order equation.

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Solving ordinary differential equations for both first order and higher-order equations typically involves finding the appropriate solution. To solve higher-order equations numerically, (*Ibragimov, N.K., 1992.*). it becomes important to convert them into a system of linear (first order) differential equations. This is a common practice in mathematics (*da Costa Campos, L.M.B.,. 2019*). Assuming that a differential equation of order n to the t h can be solved, it can be written in the following form:

$$x^{(n)} = f\left(u, x, x', x'', \dots, \frac{d^{n-1}}{du^{n-1}}\right) \qquad (1)$$

Then the system can be converted into a first-order system by this standard change of variables:

$$y_1 = x, \quad y_2 = x', \quad y_3 = x'', \dots, y_n = x^{(n-1)} = \frac{d^{n-1}x}{du^{n-1}} \qquad (2)$$

The resulting first-order system is:

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \dots, y_n^{(n)} = x^{(n)} = f(t, y_1, y_2, \dots, y_n).$$

In vector form this is simply $y' = f(u, y) = y_{i+1}\ for\ i < n$

and $f_n(u, y) = f(u, y_1, y_2, \dots, y_n)$. For example, the change of variables has the form:

$$y_1 = \alpha, y_2 = \alpha',$$

And the resulting function are given as:

$$y_1' = y_1,$$

$$y' = -ay_2 - bsin(y_1) + csin(\gamma u) \qquad (3)$$

In vector form this is:

$$y' = \begin{pmatrix} y_2 \\ -ay_2 - bsin(y_1) + csin(\gamma u) \end{pmatrix}$$

The initial conditions are given by:

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$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ u_0 \end{pmatrix} \quad (4)$$

The reason we convert a higher-order differential equation into a system of lower-order equations is to create a suitable model for solving the equation numerically (*Ibragimov, N.K., 1992*.). Most general programs for solving ODEs require us to put the ODE differential equations into a first-order linear system form. Furthermore, we make this change due to a dynamic reason in the higher order differential equation system. For example, in the case of quadratic equations, like the position of a pendulum, knowing the angle and angular velocity is necessary to understand what the pendulum is doing. We refer to the pair of values (α, α') as the state model of the system. Typically, in applications, the vector y represents the state of the system mentioned in the differential equation (*McLachlan, R.I., 1995.*)

6-1 converting Some Second- Order differential equations as first Order.

Suppose we have a second-order differential equation (where y is the undefined function and x is the variable). With luck, it is possible to transform the given equation into a first order. To solve a first-order differential equation for a function u, you can substitute u with y'. After verifying the substitution, you can solve the differential equation using any method (*Bibikov, Y.N.. 2006.*). The values of y for u can be obtained by solving the first-order differential equation, $y' = u$, which was derived from the original variable substitution. However, this approach requires some luck as defining. $u = y'$ may not always be feasible. Sometimes, we may encounter a second-order differential equation in which y is the unknown function and x is the variable. Fortunately, it's possible to transform the given equation into a first-order differential equation. To solve a first-order differential equation for a function u, simply substitute u with y'. After verifying the substitution, you can solve the differential equation using any method. The values of y for u can be obtained by solving the first-order differential equation, $y' = u$, which was derived from the original variable substitution. However, it's important to note that this approach may not always be feasible since defining u as y' may not be possible in every case (*Polyanin, A.D. and V.F. Zaitsev,. 2017*).

6-2 Solving Second-Order Differential Equations Not Explicitly Containing

To simplify differential equations, we normally set them in a way that explicitly shows $x, dy/dx$, and d^2y/dx^2, but not y. This allows us to view the differential equation as a first-order equation of dy/dx. For the sake of simplicity, we usually set it as (*da Costa Campos, L.M.B., 2019*).

$$\frac{dy}{dz} = u$$

Consequently

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(u) = \frac{du}{dx} \qquad (5)$$

The equation can be transformed by making some substitutions, resulting in a first-order differential equation for the variable u. Since the original equation doesn't involve y, neither does the differential equation for u. This provides us with an excellent opportunity to solve the equation for u using the methods that have been developed (as discussed in -(*Ascher, U.M. and L.R. Petzold, 1998: SIAM., Pucci, E. and G. Saccomandi, 2002.*). After that, we assume that 'u(z)' can be determined.

$$y(x) = \int \frac{dy}{dx}\, dz = \int u(x) dz \qquad (6)$$

When solving certain equations, it is common to end up with a formula that contains two constants. One of these constants is obtained from the general solution of the first-order differential equation in the variable u, while the other results from an integration of the variable u to obtain the values of y. We need to classify these constants as different random constants. Now, let's take a brief detour and examine the solutions to non-constant coefficient, second-order differential equations of the form (*Hartman, P., 2002, Hide, R., 1997, Kovacic, I., R. Rand, and S. Mohamed Sah, 2018*).

$$p(t)y'' + q(t)y' + r(t)y = 0$$

In general, solving differential equations with varying coefficients is often more challenging than solving those with constant coefficients. However, if we have knowledge of one solution to the differential equation, we can use a method called reduction of order to find a second solution.

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This method involves a quick look at an example to demonstrate how it works (*Bibikov, Y.N., 2006.,Pesheck, E., C. Pierre, and S. 2001*).

Example (6.1) Find the general solution to $2t^2y'' + ty' - 3y = 0, t > 0$ given that $y_1(t) = t^{-1}$

Solution: To perform a reduction of orders, it is necessary to have a known solution. Without this initial solution, it will not be possible to proceed with the reduction of order [16]. Once we have this first solution, we can assume that a second solution will take the following form:

$$y_2(t) = v(t)y_1(t) \qquad (7)$$

To find the correct value of $v(t)$, we need to choose a guess and plug it into the differential equation. Then, we can solve the new differential equation to determine the proper value of $v(t)$. For this problem, we'll need to use the form of the second solution and its derivatives. Let's get started by plugging in our guess and solving the differential equation (*Pucci, E. and G. Saccomandi, 2002.*).

$$y_2(t) = t^{-1}v, \quad y'_2(t) = -t^{-2}v + t^{-1}v'y''_2(t) = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$

When we substitute the values into the differential equation, we will get the corresponding solution.

$$2t^2(2t^{-3}v - 2t^{-2}v' + t^{-1}v'') + t(-t^{-2}v + t^{-1}v') - 3(t^{-1}v) = 0$$

After rearrangement and simplification, the expression becomes:

$$2tv'' + (-4 + 1)v' + (4t^{-1} - t^{-1} - 3t^{-1})v = 0$$

$$2tv'' - 3v' = 0$$

we note that after simplification, only the terms involving the derivatives of v remain. The term that involves v drops out. If all the calculations are done correctly, this should always happen. In some cases, like in this one, the first derivative term will also drop out. Therefore, for equation (7) to be a solution, v must satisfy certain conditions.

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$$2tv'' - 3v' = 0 \qquad (8)$$

It seems like we have a problem. To find a solution to a second order non-constant coefficient differential equation, we need to solve a different second order non-constant coefficient. However, this is not actually a problem. Since the term involving the v drops out, we can solve equation (8) using the knowledge that we already have at this point (*Bibikov, Y.N.,. 2006., Boyce, W.E., R.C. DiPrima, and D.B. 2017, Chaturantabut, S. and D.C. Sorensen2010*). We will solve this by following the steps mentioned below.: $w = v' \Longrightarrow w' = v''$. After changing the variable, equation (8) becomes $2tw' - 3w = 0$, which is a linear, first-order differential equation that can be solved. This explains why this method is called a reduction of order. By reducing a second-order differential equation to a first-order differential equation, we can solve it more easily. Since this is a simple first-order differential equation, I'll leave the details of the solving process to you. If you need a refresher on how to solve linear, first-order differential equations, feel free to ask. The solution to this differential equation is: $w(t) = ct^{(3/2)}$. However, this isn't exactly what we were looking for. We need to find a solution to equation (8). Fortunately, we can do this by recalling our change of variable, $v' = w$. Using this, we can easily solve for $v(t)$.

$$v(t) = \int w dt = \int ct^{\frac{3}{2}} dt = \frac{2}{5} ct^{\frac{5}{2}} + k$$

To obtain a second solution, we can use the most general open parenthesis 't close parenthesis possible. We have the freedom to choose the constants as per our wish to eliminate the extraneous constants. In this scenario, we can select the constants such that they clear out all the extraneous constants. $c = \frac{5}{2},\ k = 0$ Using these gives the following for $v(t)$ and the second solution.

$$v(t) = t^{\frac{5}{2}} \Rightarrow y_2(t) = t^{-1}(t^{\frac{2}{5}}) = t^{\frac{3}{2}}$$

Then general solution will then be: $y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$

If we had been given initial conditions we could then differentiate, apply the initial conditions, and solve for the constants (*Teschl, G. 2012*).

Example 6.2. Consider the second-order differential equation.

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=15e^{3x}$$

Seine

$\frac{dy}{dx}=u$ and $\frac{d^2y}{dx^2}=\frac{du}{dx}$

With the above suggestion, the differential equation will become as follows.

$$\frac{du}{dx}+u=15e^{3x}$$

This is a first-order linear differential equation with the integral factor.

$$\mu=e^{\int 2dx}=e^{2x}$$

Moving on to continue with first order linear equations,

$$e^{2x}\left(\frac{du}{dx}+u=15e^{3x}\right)$$

$$e^{2x}\frac{du}{dx}+e^{2x}u=15e^{3x}e^{2x}$$

$$\frac{d}{dx}(e^{2x}u)dx=\int 15e^{5x}\,dx$$

$$\frac{1}{2}e^{2x}u=3e^{5x}+c_0$$

There $u=2e^{-2x}(3e^{5x}+c_0)=3e^{3x}+2c_0e^{-2x}$

But $=\ dy/dx$, so the last equation can be rewritten as

$$\frac{dy}{dx}=3e^{3x}+2c_0e^{-2x}$$

which is integrated.

$$\int(3e^{3x}+2c_0e^{-2x})\,dx=e^{3x}-c_0e^{-2x}+c_1$$

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Thus (let $c_1 = -c_0/2$), then the solution to original differential equation is: $y(x) = e^{3x} - c_0 e^{-2x} + c_1$.

If the differential equation for the variable u can be separated, its solution can be obtained as shown in (*Ascher, U.M. and L.R. Petzold, 1998: SIAM., Stamenković, M., 2012*.). We may then check the constant solutions to this differential equation and consider them when integrating $y' = v$.

Remark 6.1. Our basic proposal is to apply to second-order equations in a clear and well-known way. Second-order general equations Ordinary differential equations of the form:

$$y'' = h(x, y')y' \tag{7}$$

Which we rewrite as

$$\frac{dy'}{dx} = h(x, y') \tag{8}$$

When we integrate this first-order equation, we get:

$$I_1 = g(x, y') \tag{9}$$

We invert (8) to give.

$$y' = f(x, I_1) \tag{10}$$

This form of

$$\frac{d^{n-2}}{dx} y' = f(x, I_1) \tag{11}$$

It is a trivial differential equation for $y'(x)$. Moving to the final solution, we use quadrature performance:

$$l - l_0 = \int \frac{dx}{f(x, I_1)} \tag{12}$$

Our goal is to expand the application of this method to equations of higher order, as described in reference [14].

7. Method for eliminating the differential factor

The study of systems of differential equations has a wide range of applications. In many fields, the theory of differential algebra plays a crucial role in studying solutions of ordinary differential

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equations of various orders. In this context, we are interested in eliminating the differential operator, which is a sub-theory of the algorithm through which systems of higher-order algebraic differential equations can be simplified. This is achieved by reducing the dimensions of the system of dynamic differential equations until we arrive at only one ODE variable (*Olver, P.J., 2006., Scarciotti, G. and A. Astolfi, 2015*).

7.1 Algebraic differentiation to remove differential coefficients.

Algorithmic tools can be developed to solve polynomial differential equations. However, developing these tools can be complex, and may require an algorithm for parsing. To simplify this process, an algorithm was developed based on the work of Seidenberg and Rosenfeld, incorporating references (*Boyce, W.E., R.C. DiPrima, and D.B. 2017., da Costa Campos, L.M.B. 2019., Ibragimov, N.K., 1992.*). This updated algorithm no longer requires other rules in the differential solution (*Hide, R., 1997*.). It involves eliminating differential coefficients to treat sets of higher-order polynomial differential equations where K is the differential field of coefficients ($K = R$), and U is a finite set of dependent variables.

$$\begin{cases} y_1' = a(y_2 - y_3) \\ y_2' = y_1(b - y_3) - y_2 \\ y_3' = y_1 y_2 - c y_3 \end{cases} \qquad (11)$$

The system can be rewritten as:

$$\Omega = \begin{cases} -y_1' + a(y_2 - y_3) = 0 \\ -y_2' + y_1(b - y_3) - y_2 = 0 \\ -y_3' + y_1 y_2 - c y_3 = 0 \end{cases} \qquad (12)$$

8. Reductions in n-dimensional dynamic differential equation systems

We make notes to summarize based on each system being coupled by at least one state with one variable. Otherwise, the state variables would naturally separate into lower-dimensional equations, and there would be no need for reduction.

8.1 Linear systems

For first-order linear systems of dimension n, it is possible to simplify them to a single ODE of

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higher order [9,20,24]. If the coefficient matrix of such a first-order system is full-order, then the higher-order ODE and the output will both be of order n. However, if the coefficient matrix is singular, then the resulting higher order ODE will be of order less than n (*Barao, M., J. Lemos, and R. Silva, 2002.*).

8-2 Reducible nonlinear systems and their applications

There are several possibilities for first-order nonlinear systems with dimension n, due to the formation of nonlinearities. In some cases, such as the Rossler equation, the complete differential coefficient can be omitted, and we can express all state variables in a first-order nonlinear system of higher-order ODE differential equations (*da Costa Campos, L.M.B.,. 2019, Semler, C., W. Gentleman, and M. Paıdoussis, 1996., Stamenković, M., 2012.*). However, it should be noted that the order of the individual ODE differential equations may differ from the n-dimension of the first-order system. For instance, consider the following system:

$$\begin{cases} y' = -y + x - z \\ x' = y^2 \\ z' = y - y^3 \end{cases} \qquad (13)$$

It is Clearly that the differentiation of the first equation gives:

$$\begin{cases} y'' = y' - x' - z' = y' - y^2 + y - y^3 \\ y'' - y' - y + y^2 + y^3 = 0 \end{cases} \qquad (14)$$

The following paragraph discusses an equation for one variable in the case $y(h)$ even though the original system was first-order. The system of fourth-order nonlinear dynamic differential equations is reduced to a single second-order nonlinear ODE differential equation. Even for the Lorenz equation, it will be possible to reduce this system to a single equation. The equation is not bounded by any number of differentials, and the single reduced equation of the state variable cannot be expressed as a parameter of any finite order. One state variable may satisfy a finite order of differential equations, while a different state variable may have one state variable. In such cases, nonlinearity may result in a system linearized to a distinct state variable in which the reduction to differential equations in just one variable can be made (*McLachlan, R.I., 1995.,*

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Pesheck, E., C. Pierre, and S. 2001.). The paragraph then goes on to discuss the application of the parachute equation, which is used as an idea for this application. It relies on a model of the parachutist's movement in the air using the evidence of force and air resistance together. Newton's second law states that the sum of all forces acting on an object is equal to its mass times its acceleration. When parachuting, assuming the air resistance coefficient changes between free fall and the final landing with a parachute, we can use the parachute equation in the form:

$$y'' + hy'^2 - r = 0 \qquad (15)$$

The initial conditions

$$y(0) = 0 \; and \; y'(0) = 0.$$

Here

$$h = \frac{\pi \rho C_d D^{\mu}}{8m}$$

m is the mass of the body and parachute, ρ is the density of the fluid in which the body moves, C_d is the drag coefficient for the parachute, D is the effective diameter of the parachute. The solution of equation (15) is given by:

$$y = \frac{1}{h}\left(log\left(\frac{e^{\sqrt{ghx}} + 1}{2}\right) - \sqrt{ghx}\right)$$

We must be careful to find a specific solution. This is because if we use undefined parameters, our prediction of the solution's form would be indeterminate.

$$V_p(t) = \alpha cos(\gamma t) + \beta sin(\gamma t)$$

If $\gamma_0 = \gamma$, the prediction would be problematic, as the guess for the solution is exactly the solution. In this case, we will need to add t to the solution. However, if $\gamma_0 \neq \gamma$, there will be no error in guessing. In this case, we must consider two cases. Firstly, i$\gamma_0 \neq \gamma$, the first guess is good because it will not be the complementary solution (*Bibikov, Y.N.,. 2006., Chaturantabut, S. and D.C. Sorensen2010., Teschl, G.,. 2012.)*. Upon deriving the conjecture, we can substitute it into the differential equation and simplify it to obtain the solution.

$$(-m\gamma^2\alpha + h\alpha)cos(\gamma t) + (-m\gamma^2\beta + h\beta)sin(\gamma t) = F_0 cos(\gamma t)$$

Determine the coefficients are equal:

$$cos(\gamma t): \qquad (-m\gamma^2 + h)\alpha \implies \alpha = \frac{F_0}{h - m\gamma^2}$$

$$sin(\gamma t): \qquad (-m\gamma^2 + h)\beta = 0 \implies \beta = 0$$

The particular solution is :

$$parentV_p(t) = \frac{F_0}{h - m\gamma^2}\, cos(\gamma t) = \frac{F_0}{m\left(\frac{h}{m} - \gamma^2\right)} cos(\gamma t) = \frac{F_0}{m(\gamma_0^2 - \gamma^2)} cos(\gamma t)$$

Note that we can rearrange equation (*Boyce, W.E., R.C. DiPrima, and D.B. 2017., Kovacic, I., R. Rand, and S. Mohamed Sah, 2018.*) based on desired displacement form.

$$v(t) = c_1 cos(\gamma_0 t) + c_2 sin(\gamma_0 t) + \frac{F_0}{m(\gamma_0^2 - \gamma^2)} cos(\gamma t)$$

$$v(t) = K cos(\gamma_0 t - \sigma) + \frac{F_0}{m(\gamma_0^2 - \gamma^2)} cos(\gamma t)$$

If we use the sine formula for the influence function, a similar formula can be obtained.

If $\gamma_0 = \gamma$ In this case, we will need to add t to the expectation for the solution.

$$V_p(t) = \alpha t\, cos(\gamma_0 t) + \beta t\, sin(\gamma_0 t)$$

We note that we have advanced in the solution and acknowledge that $\gamma_0 \neq \gamma$ In our expectation. It helps us acknowledge some of the simplifications we need later (*Bruno, A.D.,. 2000., Pucci, E. and G. Saccomandi, 2002., Semler, C., W. Gentleman, and M. Paıdoussis, 1996*.) Deriving our expectation, plugging it into the differential equation and then simplifying, it gives us the following:

$$(-m\gamma_0^2\alpha + h)\alpha t\, cos(\gamma t) + (-m\gamma_0^2\beta + h)\beta t\, sin(\gamma t) + 2m\gamma_0\beta\, cos(\gamma t) - 2m\gamma_0\alpha\, sin(\gamma t) = F_0 cos(\gamma t)$$

Before equating coefficients, let's recall the definition of natural frequency.

$$-m\gamma_0^2\alpha + h = -m\left(\sqrt{\frac{h}{m}}\right)^2 + h = -m\left(\frac{h}{m}\right) + h = 0$$

So, the first two terms drop out (which is a very good thing) and this gives us,

$$2m\gamma_0\, \beta cos(\gamma t) - 2m\gamma_0\alpha\, sin(\gamma t) = F_0\, cos(\gamma t)$$

Now assume that:

$$cos(\gamma t):\ 2m\gamma_0\,\beta = F_0 \Longrightarrow \beta = \frac{F_0}{2m\gamma_0}$$

$$sin(\gamma t):\ \ 2m\gamma_0\alpha = 0 \Longrightarrow \alpha = 0$$

In this case we be:

$$V_p(t) = \frac{F_0}{2m\gamma_0}t\ sin(\gamma t)$$

The displacement for this case is given:

$$v(t) = c_1cos(\gamma_0 t) + c_2sin(\gamma_0 t) + \frac{F_0}{2m\gamma_0}t\ sin(\gamma_0 t)$$

$$v(t) = Kcos(\gamma_0 t - \sigma) + \frac{F_0}{2m\gamma_0}t\ sin(\gamma_0 t)$$

Depending on the above model we find, what is the purpose of the two cases in this case? In the first case, $\gamma_0 \neq \gamma$ The displacement function consists of two cosines and is well always behaved. In contrast, in the second case, $\gamma_0 = \gamma$ We will have some issues in t increases (*Barao, M., J. Lemos, and R. Silva, 2002., Chaturantabut, S. and D.C. Sorensen. 2009., Shakeri, F. and M. Dehghan, 2008.*). Add The presence of t in the solution means that we will see an oscillation that increases in amplitude as t increases. This condition is called air friction, and we generally want to avoid that. Aassuming that the impact function was:

$$F(t) = F_0\cos(\gamma_0 t)$$

We will also have the possibility of vibration if we assume the impact function as follows:

$$F(t) = F_0\sin(\gamma_0 t)$$

It is important to note that we should not assume that the influence function will always take one of the two forms mentioned earlier. Coercive jobs can come in various forms, and if we come across an effect function that differs from the one used in this case, we will need to deal with undefined coefficients or different parameters to find a solution (*Boyce, W.E., R.C. DiPrima, and D.B. 2017., Chaturantabut, S. and D.C. Sorensen. 2009, Kovacic, I., R. Rand, and S. Mohamed Sah, 2018.*).

9. Possibility of Numerical Solution

We can reduce any higher order ordinary differential equation to a system of first order ordinary differential equations by substitution. Let's assume second order ordinary differential equations and consider the initial value of the vector y.

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad (15)$$

Nonlinearity can have several effects. Resonance can cause an increase in the amplitude of motion, which, in turn, can affect the relationship between period and amplitude. A more realistic model can be used to obtain a tendency to produce large movements. These concepts are discussed in sources such as (*Bibikov, Y.N.,. 2006., Boyce, W.E., R.C. DiPrima, and D.B. 2017, Olver, P.J., 2006*).

$$\frac{d^2y}{d\phi^2} + \left(\alpha - 2qcos(2\phi)\right)y = 0 \quad (16)$$

By solving a system of first-order differential equations, one can reduce higher-order differential equations to this form.

Example 9.1. Find the solution to the following:

$$zy' - 2y = z^5\sin(2z) - z^3 + 4z^4, \qquad y(\pi) = \frac{3}{2}\pi^2$$

Solution First, divide through by t to get the differential equation in the correct form:

$$y' - \frac{2}{z}y = z^4\sin 2z - z^2 + 4z^3$$

Now that we have done this, we can find the integrating factor, $\beta(z)$

$$\beta(z) = e^{\int -\frac{2}{z}dz} = e^{-2ln|z|}$$

It's important to remember that the minus sign is a part of $\beta(z)$. Omitting this minus sign can transform a simple problem into a very difficult or even impossible one. Therefore, it's crucial to be careful while solving such problems. Moving on to the next step, we can simplify this problem just like we did in the previous example, (*Barao, M., J. Lemos, and R. Silva, 2002., Bruno, A.D.,. 2000.*).

$$\beta(z) = e^{-2ln|z|} = e^{ln|z|^{-2}} = |z|^{-2} = z^{-2}$$

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Since we are squaring the term, we can drop the absolute value bars. Then, multiply the differential equation by the integrating factor (using the rewritten one, not the original equation).

$$(z^{-2}y)' = z^2\sin(2z) - 1 + 4z$$

Integrate both sides and solve for the solution.

$$z^{-2}y(z) = \int z^2\sin(2z)\,dz + \int -1 + 4z\,dz$$

$$z^{-2}y(z) = -\frac{1}{2}z^2\cos(2z) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2z) - t + 2z^2 + c$$

$$y(z) = -\frac{1}{2}z^4\cos(2z) + \frac{1}{2}z^3\sin(2z) + \frac{1}{4}z^2\cos(2z) - z^3 + 2z^4 + cz^2$$

Apply the initial condition to find the value of *c*.

$$\frac{3}{2}\pi^4 = y(\pi) = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2$$

$$\pi^3 - \frac{1}{4}\pi^4 = c\pi^2, \qquad c = \pi - \frac{1}{4}$$

Then the solution is:

$$y(z) = -\frac{1}{2}z^4\cos(2z) + \frac{1}{2}z^3\sin(2z) + \frac{1}{4}z^2\cos(2z) - z^3 + 2z^4 + \left(\pi - \frac{1}{4}\right)z^2$$

Then the plot of the solution is:

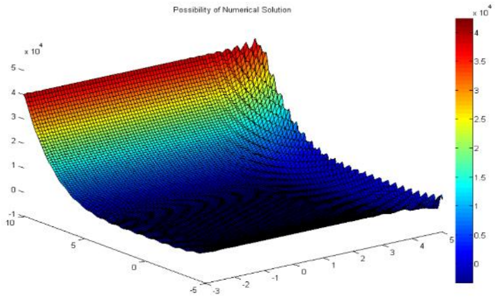

Figure 1

10. Classical Mathieu's Equation: Mechanical Models and Applications

The concept of geometric nonlinearity is significant in the study of the Mathieu equation. Resonance between the forcing frequency and the unforced natural frequency of the oscillator can result in infinite solutions, as shown in Figure 2. However, nonlinear systems have

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limitations, unlike linear systems. In the case of a vertically driven pendulum, the nonlinear terms in the Mathieu equation can be included by increasing the sin x expansion of equation. (16). Conclusions from this analysis were drawn in reference.

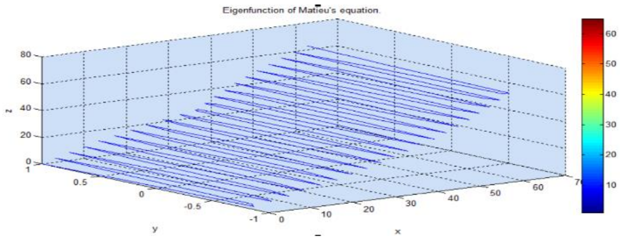

Figure 2

In this section, we briefly review some known facts about the solutions of the Mathieu equation (*Hartman, P., 2002*). The Mathieu equation is a second-order homogeneous linear differential equation of the form in a Taylor series.

$$\frac{d^2y}{d\phi^2} + \big(\alpha - 2qcos(2\phi)\big)y = 0$$

The constants q are often referred to as the characteristic element and the parameter in parabola. Equation (16) deals only with real values of alpha and q, although we can also consider the more general case where alpha and q are complex numbers. By replacing the independent variable $\emptyset \rightarrow i\emptyset$ in parabola (15), we obtain the modified Mathieu equation.

$$\frac{d^2y}{d\phi^2} + \big(\alpha - 2qcosh(2\phi)\big)y = 0 \qquad (17)$$

$\phi^{¼} = 0\, or \phi^{¼} = p$ The deflection angle of a pendulum is represented by the generalized coordinate g. The gravitational acceleration is denoted by L and the vertical motion of the continuous support is represented by $A\ cos(\emptyset)$. The equilibrium solutions are given by

$\phi^{¼} = 0\, or \phi^{¼} = p$. To check stability, we can linearize the equation. Equation (16) deals with balance and stability and can be derived in the form of an equivalent equation, as shown in equation (10). If the support movement is defined by $Acos(\beta\emptyset)$, then the motion for less phi is represented by the numerator.

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$$\frac{d^2y}{d\phi^2}+\left(\omega_0^2-\frac{\mathrm{Ag}^2}{L}cos(\mathrm{g}\phi)\right)y=0 \qquad (18)$$

In this study, the effect of damping on the transition curves of the Mathieu equation was analyzed by applying the two-variable expansion method to the damper Mathieu's equation.

$$\frac{d^2y}{d\phi^2}+k\frac{dy}{d\phi}+(\mu+\delta cos\phi)y=0 \qquad (19)$$

To apply the perturbation method, the damping coefficient k was rescaled accordingly.

$$\frac{d^2y}{d\phi^2}+\big(\alpha-2qcosh(2\phi)\big)y=0$$

The Mathieu equation, as expressed in its classical form (15), is typically used to solve differential equations in two scenarios: Case 1 - for systems that involve a periodic effect, and Case 2 - for stability studies of periodic motions in nonlinear autonomous systems (*Ascher, U.M. and L.R. Petzold, 1998: SIAM.*).

10.1 Algorithm for Nonlinear Higher-Order Differential Equations

Algorithm 1 Nonlinear Higher-Order Differential Equations (NHODE)

```
% solving Boundary value problem with unknown parameter
% Example: Mathieu's Equation; y''+(Lambda-2.q.cos(2x).y=0
% y (0)=1; y'(0)=0; y'(pi)=0;
% The task is to compute the fourth (q = 5) eigenvalue lambda of Mathieu's equation
% solution Initial guess
% Initial guess of the unknown parameter
% Solve the problem using bvp4c
% sol= bvp5c(@mat4ode, @mat4bc,solinit):
% --------------------------------------------------------------
% Equations to solve
% y''+(lambda-2.q.cos(2x). y=0
% -------------------------------------------------------------------------
% conditions
% y'(0) = 0; --> ya(2)    initial condition
```

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```
% y'(pi)=0; --> yb(2)    initial condition
% y(0)=1; --> ya(1)-1     initial condition
% Note ya(1) is a condition of y(0) , yb(1) is a condition of y(tf)
%       ya(2) is a condition of y'(0), yb(2) is a condition of y'(tf)
```

Columns 71 through 80

44.42 66	45.06 12	45.69 59	46.33 06	46.96 52	47.59 99	48.23 46	48.86 92	49.50 39	50.13 85

Columns 81 through 90

50.77 32	51.40 79	52.04 25	52.67 72	53.31 19	53.94 65	54.58 12	55.21 59	55.85 05	56.48 52

Columns 91 through 100

57.11 99	57.75 45	58.38 92	59.02 39	59.65 85	60.29 32	60.92 79	61.56 25	62.19 72	62.83 19

11. Conclusions and recommendations

The system of nonlinear differential equations was transformed into simple linear differential equations, and the Matthew equation was described in a discrete way. The time averages of the periodic coefficients in the nonlinear differential equation were calculated after reducing it to a linear degree, resulting in a differential equation with constant coefficients that is easier to solve. The computer was used to process the equation and calculate approximate solutions using MATLAB. The study of the Matthew equation is of great importance for several reasons: It helps in understanding many complex physical phenomena, such as resonance and vibrations that use nonlinear differential equations. The results of the study of the equation can be used in designing more efficient and stable control systems. It helps in predicting the behavior of complex systems, which contributes to improving their design and maintenance. The Matthew equation is a powerful tool for understanding and analyzing many physical and engineering phenomena. Despite the challenges facing its study, ongoing research contributes to the development of new methods for solving and analyzing it, which opens up new horizons for applications in various fields.

Ethics statements

High standards of ethical behavior in citation and honesty in all scholarly reporting are maintained. Meet the high standards of the research conflict of interest policy Conflict of Interests The authors declare that there is no conflict of interest regarding the publication of this paper.

Conflict of Interests

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